REAL AND COMPLEX ANALYSIS

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CHAPTER ONE

ABSTRACT INTEGRATION

1 Does there exist an infinite σ-algebra which has only countable many members?

Solution: No. Suppose $\mathfrak{M}$ be a σ-algebra on X which has countably infinite members. For each $x \in X$ define $B_x = \cap_{x \in M \in \mathfrak{M}} M$. Since $\mathfrak{M}$ has countable members, so the intersection is over countable members or less, and so B_x belongs $\mathfrak{M}$, since $\mathfrak{M}$ is closed under countable intersection. Define $\mathfrak{N} = \{B_x \mid x \in X\}$. So $\mathfrak{N} \subset \mathfrak{M}$. Also we claim if $A, B \in \mathfrak{N}$, with $A \neq B$, then $A \cap B = \emptyset$. Suppose $A \cap B \neq \emptyset$, then some $x \in A$ and same $x \in B$, but that would mean $A = B = \cap_{x \in M \in \mathfrak{M}} M$. Hence $\mathfrak{N}$ is a collection of disjoint subsets of X. Now if cardinality of $\mathfrak{N}$ is finite say $n \in \mathbb{N}$, then it would imply cardinality of $\mathfrak{M}$ is 2^n, which is not the case. So cardinality of $\mathfrak{N}$ should be at least $\aleph_0$. If cardinality of $\mathfrak{N} = \aleph_0$, then cardinality of $\mathfrak{M} = 2^{\aleph_0} = \aleph_1$, which is not possible as $\mathfrak{M}$ has countable many members. Also if cardinality of $\mathfrak{N} \geqslant \aleph_1$, so is the cardinality of $\mathfrak{M}$, which again is not possible. So there does not exist an infinite σ-algebra having countable many members. ∎

2 Prove an analogue of Theorem 1.8 for n functions.

Solution: Analogous Theorem would be: Let $u_1, u_2, \ldots, u_n$ be real-valued measurable functions on a measurable space X, let Φ be a continuous map from $\mathbb{R}^n$ into topological space Y, and define

$$h(x) = \Phi(u_1(x), u_2(x), \ldots, u_n(x))$$

for $x \in X$. Then $h : X \to Y$ is measurable.

Proof: Define $f : X \to \mathbb{R}^n$ such that $f(x) = (u_1(x), u_2(x), \ldots, u_n(x))$. So

$h = \Phi \circ f$. So using Theorem 1.7, we only need to show f is a measurable function. Consider a cube Q in $\mathbb{R}^n$. $Q = I_1 \times I_2 \times \cdots \times I_n$, where I_i are the intervals in $\mathbb{R}$. So

$$f^{-1}(Q) = u_1^{-1}(I_1) \cap u_2^{-1}(I_2) \cap \cdots \cap u_n^{-1}(I_n)$$

Since each u_i is measurable, so $f^{-1}(Q)$ is measurable for all cubes $Q \in \mathbb{R}^n$. But every open set V in $\mathbb{R}^n$ is a countable union of such cubes, i.e $V = \cup_{i=1}^{\infty} Q_i$, therefore

$$f^{-1}(V) = f^{-1}(\bigcup_{i=1}^{\infty} Q_i) = \bigcup_{i=1}^{\infty} f^{-1}(Q_i)$$

Since countable union of measurable sets is measurable, so $f^{-1}(V)$ is measurable. Hence f is measurable. ∎

3 Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is a measurable for every rational r, then f is measurable.

Solution: Let $\mathfrak{M}$ denotes the σ-algebra of measurable sets in X. Let Ω be the collections of all $E \subset [-\infty, \infty]$ such that $f^{-1}(E) \in \mathfrak{M}$. So for all rationals r, $[r, \infty] \in \Omega$. Let $\alpha \in \mathbb{R}$; we will show $(\alpha, \infty] \in \Omega$; hence from Theorem 1.12(c) conclude that f is measurable.

Since rationals are dense in R, therefore there exists a sequence of rationals $\{r_i\}$ such that $r_i > \alpha$ and $r_i \to \alpha$. Also $(\alpha, \infty] = \bigcup_1^{\infty} [r_i, \infty]$. Each $[r_i, \infty] \in \Omega$ and Ω is a σ-algebra (Theorem 1.12(a)) and hence closed under countable union; therefore $(\alpha, \infty] \in \Omega$. And so from Theorem 1.12(c), we conclude f is measurable. ∎

4 Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, prove the following assertions:

(a)
$$\limsup_{x \to \infty} (-a_n) = -\liminf_{n \to \infty} a_n.$$

(b)
$$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$$
provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all n, then
$$\limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n.$$

Show by an example that the strict inequality can hold in (b).

Solution: (a) We have for all $n \in \mathbb{N}$,

$$\sup_{i\geq n} \{-a_i\} = -\inf_{i\geq n} \{a_i\}$$

taking limit $n \to \infty$, we have desired equality.

(b) Again for all $n \in \mathbb{N}$, we have

$$\sup_{i\geq n}\{a_i + b_i\} \leq \sup_{i\geq n}\{a_i\} + \sup_{i\geq n}\{b_i\}$$

Taking limit $n \to \infty$, we have desired inequality.

(c) Since $a_n \leq b_n$ for all n, so for all n we have

$$\inf_{i\geq n}\{a_i\} \leq \inf_{i\geq n}\{b_i\}$$

Taking limit $n \to \infty$, we have desired inequality.

For strict inequality in (b), consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. ∎

5 (a) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x \ : \ f(x) < g(x)\}, \{x \ : \ f(x) = g(x)\}$$

are measurable.

Solution: Given f, g are measurable, therefore from 1.9(c) we conclude $g-f$ is also measurable. But then $\{x \mid f(x) < g(x)\} = (g-f)^{-1}(0, \infty]$ is a measurable set by Theorem 1.12(c).

$$\begin{aligned} \text{Also } \{x \mid f(x) = g(x)\} &= (g-f)^{-1}(0) \\ &= (g-f)^{-1}\left(\bigcap\left(-\frac{1}{n}, \frac{1}{n}\right)\right) \\ &= \bigcap (g-f)^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right) \end{aligned}$$

Since each $(g-f)^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$ is measurable, so is their countable intersection. Hence $\{x \mid f(x) = g(x)\}$ is measurable. ∎

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Solution: Let f_i be the sequence of real-measurable functions. Let A denotes the set of points at which f_i converges to a finite limit. But then

$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{i,j \geqslant m} \{x \mid |f_i(x) - f_j(x)| < \frac{1}{n}\} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{i,j \geqslant m} (f_i - f_j)^{-1} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

Since for each i, j, $f_i - f_j$ is measurable, so $(f_i - f_j)^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$ is measurable too for all n. Also countable union and intersection of measurable sets is measurable, we conclude A is measurable. $\blacksquare$

6 Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case and $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

Solution: *$\mathfrak{M}$ is a σ-algebra in X*: $X \in \mathfrak{M}$, since $X^c = \emptyset$ is countable. Similarly $\emptyset \in \mathfrak{M}$. Next if $A \in \mathfrak{M}$, then either A or A^c is countable, that is either $(A^c)^c$ is countable or A^c is countable; showing $A^c \in \mathfrak{M}$. So $\mathfrak{M}$ is closed under complement. Finally, we show $\mathfrak{M}$ is closed under countable union. Suppose $A_i \in \mathfrak{M}$ for $i \in \mathbb{N}$, we will show $\bigcup A_i$ also belongs to $\mathfrak{M}$. If all A_i are countable, so is their countable union, so $\bigcup A_i \in \mathfrak{M}$. But when all A_i are not countable means at least one say A_j is uncountable. Then A_j^c is countable. Also $(\bigcup A_i)^c \subset A_j^c$, showing $(\bigcup A_i)^c$ is countable. So $\bigcup A_i \in \mathfrak{M}$. Hence $\mathfrak{M}$ is closed under countable union.

μ is a measure on $\mathfrak{M}$: Since μ takes values 0 and 1, therefore $\mu(A) \in [0, \infty]$ for all $A \in \mathfrak{M}$. Next we show μ is countable additive. Let A_i for $i \in \mathbb{N}$ are disjoint measurable sets. Define $A = \bigcup A_i$. We will show $\mu(A) = \sum \mu(A_i)$. If all A_i are countable, so is A; therefore $\mu(A_i) = 0$ for all i and $\mu(A) = 0$; and the equation $\mu(A) = \sum \mu(A_i)$ holds good. But when all A_i are not countable means at least one say A_j is uncountable. Since $A_j \in \mathfrak{M}$, therefore A_j^c is countable. Also Since all A_i are disjoint, so for $i \neq j$, $A_i \in A_j^c$. So $\mu(A_i) = 0$ for $i \neq j$. Also $\mu(A_j) = \mu(A) = 1$ since both are uncountable. Hence $\mu(A) = \sum \mu(A_i)$.

Characterization of measurable functions and their integrals: Assume functions are real valued. First we isolate two class of measurable functions denoted by F_∞ and $F_{-\infty}$, defines as:

$$F_\infty = \{f \mid f \text{ is measurable } \& \ f^{-1}([-\infty, \alpha]) \text{ is countable for all } \alpha \in \mathbb{R}\}$$

$$F_{-\infty} = \{f \mid f \text{ is measurable } \& \ f^{-1}([\alpha, \infty]) \text{ is countable for all } \alpha \in \mathbb{R}\}$$

Next we characterize the reaming measurable functions. Since $f \notin F_\infty$ or $F_{-\infty}$, therefore $f^{-1}([\alpha, \infty])$ is uncountable for some $\alpha \in \mathbb{R}$. Therefore α_f defined as $\sup\{\alpha \mid f^{-1}([\alpha, \infty]) \text{ is countable}\}$ exists. So if $\beta > \alpha_f$, then $f^{-1}([\beta, \infty])$ is countable. Also if $\beta < \alpha_f$, then $f^{-1}([-\infty, \beta]) = X - f^{-1}((\beta, \infty])$. Since $f^{-1}((\beta, \infty])$ is uncountable and belongs to $\mathfrak{M}$, therefore $X - f^{-1}((\beta, \infty])$ is countable. And so $f^{-1}(\alpha_f)$ is uncountable. Also $f^{-1}(\alpha_f) \in \mathfrak{M}$, therefore $f^{-1}(\gamma)$ is countable for all $\gamma \neq \alpha_f$. Thus if f is a measurable function then either $f \in F_\infty or F_{-\infty}$, or there exists $\alpha_f \in \mathbb{R}$ such that $f^{-1}(\alpha_f)$ is uncountable while $f^{-1}(\beta)$ is countable for all $\beta \neq \alpha_f$. Once we have characterization, integrals are easy to describe:

$$\int_X f \, d\mu = \begin{cases} \infty & \text{if } f \in F_\infty \\ -\infty & \text{if } f \in F_{-\infty} \\ \alpha_f & \text{else} \end{cases} \qquad \blacksquare$$

7 Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots$, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

Solution: Take $g = f_1$ in the Theorem 1.34 to conclude

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

For showing $f_1 \in L^1(\mu)$ is a necessary condition for the conclusion, take $X = \mathbb{R}$ and $f_n = \chi_{[n,\infty)}$. So we have $f(x) = 0$ for all x, and therefore $\int_X f \, d\mu = 0$. While $\int_X f_n \, d\mu = \infty$ for all n. $\blacksquare$

8 Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

Solution: With the described sequence of f_n, strict inequality occurs in Fatou's Lemma (1.28). We have

$$\int_X \left(\liminf_{n\to\infty} f_n \right) \, d\mu = 0$$

While

$$\liminf_{n\to\infty} \int_X f_n \, d\mu = \min(\mu(E), \mu(X) - \mu(E)) \neq 0,$$

assuming $\mu(X) \neq \mu(E)$. ■

9 Suppose μ is a positive measure on X, $f : X \to [0, \infty]$ is measurable, $\int_X f \, d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\to\infty} \int_X n \log[1 + (f/n)^\alpha] \, d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

Hint: If $\alpha \geq 1$, prove that the integrands is dominated are dominated by αf. If $\alpha < 1$, Fatou's lemma can be applied.

Solution: As given in the hint, we consider two cases for α:
Case when $0 < \alpha < 1$: Define $\phi_n(x) = n \log(1 + (f(x)/n)^\alpha)$. Since $\phi_n : X \to [0, \infty]$, therefore Fatou's lemma is applicable. So

$$\int_X (\liminf_{n\to\infty} \phi_n) \, d\mu \leqslant \liminf_{n\to\infty} \int_X \phi_n \, d\mu$$

Also

$$\lim_{n\to\infty} n \log(1 + (f(x)/n)^\alpha) = \lim_{n\to\infty} \frac{\frac{1}{1+(f(x)/n)^\alpha} \frac{-\alpha f(x)^\alpha}{n^{\alpha+1}}}{\frac{-1}{n^2}} = \frac{\alpha n^{1-\alpha} f(x)^\alpha}{1 + (f(x)/n)^\alpha}$$

Since $\alpha < 1$ and $\int_X f \, d\mu < \infty$, therefore

$$\lim_{n\to\infty} n \log(1 + (f(x)/n)^\alpha) = \infty \text{ a.e } x \in X$$

And hence $\int_X(\liminf_{n\to\infty}\phi_n)\,d\mu=\infty$. Therefore

$$\lim_{n\to\infty}\int_X \phi_n\,d\mu \geqslant \liminf_{n\to\infty}\int_X \phi_n\,d\mu = \infty$$

Case when $\alpha \geqslant 1$: We claim $\phi_n(x)$ is dominated by $\alpha f(x)$. For a.e. $x\in X$ and $\alpha\geqslant 1$, we need to show

$$n\log(1+(f(x)/n)^\alpha)\leqslant \alpha f(x) \text{ for all } n$$
$$\text{i.e. } \log\left(1+\left(\frac{f(x)}{n}\right)^\alpha\right)\leqslant \alpha\frac{f(x)}{n} \tag{1}$$

Define $g(\lambda)=\log(1+\lambda^\alpha)-\alpha\lambda$ for $\lambda\geqslant 0$. So if $g(\lambda)\leqslant 0$ for $\alpha\geqslant 1$ and $\lambda\geqslant 0$, then (1) follows by taking $\lambda=f(x)/n$. So we need show $g(\lambda)\leqslant 0$ for $\alpha\geqslant 1$ and $\lambda\geqslant 0$. Computing derivative of g, we have

$$g'(\lambda)=-\frac{\alpha(1+\lambda^\alpha-\lambda^{\alpha-1})}{1+\lambda^\alpha}$$

When $0\geqslant\lambda\geqslant 1$, we have $1-\lambda^{\alpha-1}\geqslant 0$; while when $\lambda>1$, we have $\lambda^\alpha-\lambda^{\alpha-1}>0$. Thus $g'(\lambda)\leqslant 0$. Also $g(0)=0$, therefore $g(\lambda)\leqslant 0$ for all $\lambda\geqslant 0$ and $\alpha\geqslant 1$. And so for $\alpha\geqslant 1$ we have $\log(1+(f(x)/n)^\alpha)\leqslant\alpha f(x)$ for all n and a.e $x\in X$. Since $\alpha f(x)\in L^1(\mu)$, DCT(Theorem 1.34) is applicable. Thus

$$\lim_{n\to\infty}\int_X n\log(1+(f/n)^\alpha)\,d\mu=\int_X\lim_{n\to\infty}(n\log(1+(f/n)^\alpha))\,d\mu$$

When $\alpha=1$, $\lim_{n\to\infty}(n\log(1+(f/n)^\alpha))=f(x)$ (calculating the same as calculated for the case $\alpha<1$). And when $\alpha>1$, we have $\lim_{n\to\infty}(n\log(1+(f/n)^\alpha))=0$. And hence

$$\lim_{n\to\infty}\int_X n\log(1+(f/n)^\alpha)\,d\mu=\begin{cases}\infty & \text{if } 0<\alpha<1,\\ c & \text{if } \alpha=1,\\ 0 & \text{if } 1<\alpha<\infty.\end{cases} \qquad \blacksquare$$

10 Suppose $\mu(X)<\infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n\to f$ uniformly on X. Prove that

$$\lim_{n=\infty}\int_X f_n\,d\mu=\int_X f\,d\mu,$$

and show that the hypothesis "$\mu(X) < \infty$" cannot be omitted.
Solution: Let $\epsilon > 0$. Since $f_n \to f$ uniformly, therefore there exists $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geqslant n_0$$

Therefore $|f(x)| < |f_{n_0}(x)| + \epsilon$. Also $|f_n(x)| < |f(x)| + \epsilon$. Combining both equations, we get

$$|f_n(x)| < |f_{n_0}| + 2\epsilon \quad \forall\, n \geqslant n_0$$

Define $g(x) = \max(|f_1(x)|, \cdots, |f_{n_0-1}(x)|, |f_{n_0}(x)| + 2\epsilon)$, then $f_n(x) \leqslant g(x)$ for all n. Also g is bounded. Since $\mu(X) < \infty$, therefore $g \in L^1(\mu)$. Now apply DCT(Theorem 1.34) to get

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu$$

To show "$\mu(X) < \infty$" is a necessary condition, consider $X = \mathbb{R}$ with usual measure μ, and $f_n(x) = \frac{1}{n}$. We have $\lim_{n\to\infty} \int_X f_n\, d\mu = \infty$, while $\int_X f\, d\mu = 0$, since $f = 0$.

REMARK: The condition "$f_n \to f$ uniformly" is also a necessary condition. ∎

11 Show that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

in Theorem 1.41, and hence prove the theorem without any reference to integration.
Solution: A is defined as the collections of all x which lie in infinitely many E_k. Thus $x \in A \iff x \in \bigcup_{k=n}^{\infty} E_k \quad \forall\, n \in \mathbb{N}$; and so

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

Now let $\epsilon > 0$. Since $\sum_{k=1}^{\infty} \mu(E_k) < \infty$, therefore there exists $n_0 \in \mathbb{N}$ such that $\sum_{k=n_0}^{\infty} \mu(E_k) < \epsilon$. And

$$\begin{aligned}\mu(A) &= \mu(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) \\ &\leqslant \mu(\bigcup_{k=n_0}^{\infty} E_k) \\ &\leqslant \sum_{k=n_0}^{\infty} \mu(E_k) \\ &< \epsilon\end{aligned}$$

Make $\epsilon \to 0$ to conclude $\mu(A) = 0$. ▮

12 Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

Solution: Let $(X, \mathfrak{M}, \mu)$ be the measure space. Suppose the statement is not true. Therefore there exists a $\epsilon > 0$ such that there exists no $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ whenever $\mu(E) < \delta$. That means for each $\delta > 0$, there exists a $E_\delta \in \mathfrak{M}$ such that $\mu(E_\delta) < \delta$, while $\int_{E_\delta} |f| \, d\mu > \epsilon$. By taking $\delta = 1/2^n$, where $n \in \mathbb{N}$, we construct a sequence of measurable sets $\{E_{1/2^n}\}$, such that $\mu(E_{1/2^n}) < 1/2^n$ for all n and $\int_{E_{1/2^n}} |f| \, d\mu > \epsilon$.

Now define $A_k = \bigcup_{n=k}^{\infty} E_{1/2^n}$ and $A = \bigcap_{k=1}^{\infty} A_k$. We have $A_1 \supset A_2 \supset A_3 \cdots$, and $\mu(A_1) = \mu(\bigcup_{n=1}^{\infty} E_{1/2^n}) \leqslant \sum_{n=1}^{\infty} \mu(E_{1/2^n}) < \sum_{n=1}^{\infty} 1/2^n < \infty$. Therefore from Theorem 1.19(e), we conclude $\mu(A_k) \to \mu(A)$.

Next define $\phi : \mathfrak{M} \to [0, \infty]$ such that $\phi(E) = \int_E |f| \, d\mu$. Clearly, by Theorem 1.29, ϕ is a measure on $\mathfrak{M}$. Therefore, again by Theorem 1.19(e), we have $\phi(A_k) \to \phi(A)$. Since $A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_{1/2^n}$, therefore from previous Exercise, we get $\mu(A) = 0$. Therefore $\phi(A) = \int_A |f| \, d\mu = 0$. While

$$\phi(A_k) = \phi(\bigcup_{n=k}^{\infty} E_{1/2^n}) \geqslant \phi(E_{1/2^k}) = \int_{E_{1/2^k}} |f| \, d\mu > \epsilon$$

Therefore $\phi(A_k) \nrightarrow \phi(A)$, a contradiction. Hence the result. ▮

13 Show that proposition 1.24(c) is also true when $c = \infty$.

Solution: We have to show

$$\int_X cf\,d\mu = c\int_X f\,d\mu, \text{ when } c = \infty \text{ and } f \geq 0$$

We consider two cases: when $\int_X f\,d\mu = 0$ and $\int_X f\,d\mu > 0$. When $\int_X f\,d\mu = 0$, we have from Theorem 1.39(a), $f = 0$ a.e., therefore, $cf = 0$ a.e.. And hence

$$\int_X cf\,d\mu = 0 = c\int_X f\,d\mu$$

While when $\int_X f\,d\mu > 0$, there exist a $\epsilon > 0$ and a measurable set E, such that $\mu(E) > 0$ and $f(x) > \epsilon$ whenever $x \in E$; because otherwise $f(x) < \epsilon$ a.e. for all $\epsilon > 0$; making $\epsilon \to 0$, we get $f(x) = 0$ a.e. and hence $\int_X f\,d\mu = 0$, which is not the case. But then

$$\int_X cf\,d\mu \geqslant \int_E cf\,d\mu > \epsilon\int_E c\,d\mu = \infty$$

$$\text{Also } c\int_X f\,d\mu = \infty$$

Hence the proposition is true for $c = \infty$ too. ∎

CHAPTER

TWO

POSITIVE BOREL MEASURES

1 Let $\{f_n\}$ be s sequence of real nonnegative functions on R^1, consider the following four statements:
(a) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.
(b) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is upper semicontinuous.
(c) If each f_n is upper semicontinuous, then $\sum_1^\infty f_n$ is upper semicontinuous.
(c) If each f_n is lower semicontinuous, then $\sum_1^\infty f_n$ is lower semicontinuous.
Show that three of these are true and that one is false. What happens if the word "nonnegative" is omitted? Is the truth of the statements affected if R^1 is replaced by a general topological space?

Solution: (a) For $\alpha \in R^1$, we have

$$\{x \mid (f_1 + f_2)(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} (\{x \mid f_1(x) < r\} \cap \{x \mid f_2(x) < \alpha - r\})$$

And since f_1 and f_2 are upper semicontinuous; and countable union of open sets is open, therefore we conclude $\{x \mid (f_1 + f_2)(x) < \alpha\}$ is an open set for all $\alpha \in \mathbb{R}$. Hence $f_1 + f_2$ is upper semicontinuous.

(b) Again for $\alpha \in R^1$, we have

$$\{x \mid (f_1 + f_2)(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} (\{x \mid f_1(x) > r\} \cap \{x \mid f_2(x) > \alpha - r\})$$

Therefore lower semicontinuity of f_1 and f_2 implies $f_1 + f_2$ is also lower semicontinuous.

(c) It may not be true in general. We give a counterexample. Define $f_n = \chi_{[\frac{1}{n+1},\frac{1}{n}]}$. So each f_n is upper semicontinuous (2.8(b)). Also $f = \sum f_n = \chi_{(0,1]}$, which is not a upper semicontinuous function.

(d) Define $s_k = \sum_{n=1}^{k} f_n$. We have for $\alpha \in \mathbb{R}$,

$$\{s_k > \alpha\} = \bigcup_{r_1,r_2,\cdots,r_{k-1}\in\mathbb{Q}} (\{f_1 > r_1\} \cap \{f_2 > r_2\} \cap \cdots \cap \{f_k > \alpha - r_{k-1}\})$$

Each f_n being lower semicontinuous implies s_k is lower semicontinuous for all k. But then 2.8(c) implies $\sup_k s_k$ is also lower semicontinuous. Since $s_k(x)$ is increasing for all x, we have $\sup_k s_k = \lim_{k\to\infty} s_k = \sum_{n=1}^{\infty} f_n$. Hence $\sum_{n=1}^{\infty} f_n$ is lower semicontinuous.

If the word "nonnegative" is omitted: Clearly (a) and (b) remain unchanged. Also the counterexample of (c) is still valid as a counterexample. But in (d), now $\sup_k s_k$ may not be equal to $\lim_{k\to\infty} s_k$. We provide a concrete counterexample. Define

$$f_n = \begin{cases} \chi_{(-1,1)} & \text{if } n = 1, \\ -\chi_{[\frac{1}{n+1},\frac{1}{n}]} & \text{if } n \geqslant 2 \end{cases}$$

Easy to check that each f_n is lower semicontinuous and that $\sum_1^{\infty} f_n = \chi_{(-1,0]} + \chi_{(0.5,1)}$; which obviously is not lower semicontinuous.

If R^1 is replaced by general topological space: Domain of f_n is incidental, all we used in proving (a)-(d) are the two facts that rationals are dense in the range of f_n; and supremum of lower semicontinuous functions is lower semicontinuus. So as long as f_n are real-valued functions, results will remain the same. ∎

2 Let f be an arbitrary complex function on R^1, and define

$$\phi(x, \delta) = \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\},$$

$$\phi(x) = \inf\{\phi(x, \delta) : \delta > 0\}.$$

Prove that ϕ is upper semicontinuous, that f is continuous at a fixed point x is and only if $\phi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is G_δ.

Formulate and prove an analogous statement for general topological spaces in place of R^1.

Solution: *ϕ is upper semicontinuous*: First note that $\phi(x, \delta_1) \leqslant \phi(x, \delta_2)$ whenever $\delta_1 < \delta_2$. Therefore

$$\phi(x) = \inf\{\phi(x, \delta) \mid \delta > 0\} = \lim_{\delta \to 0} \phi(x, \delta)$$

For $\alpha \in R^1$, consider $\{x \mid \phi(x) < \alpha\} = V_\alpha$(say). We need to show V_α is open for all $\alpha \in R^1$, for proving ϕ is upper semicontinuous. If $V_\alpha = \emptyset$, then clearly it is an open set. If $V_\alpha \neq \emptyset$, then let some $x_0 \in V_\alpha$. Therefore $\phi(x_0) < \alpha$. But $\phi(x_0) = \lim_{\delta \to 0} \phi(x_0, \delta)$. Therefore, there exist a $\delta_0 > 0$, such that $\phi(x_0, \delta_0) < \alpha$. Now consider the open ball B around x_0 of radius $\delta_0/2$, i.e $B = (x_0 - \delta_0/2, x_0 + \delta_0/2)$. If $y \in B$, then

$$\begin{aligned}
\phi(y) \leqslant \phi(y, \delta_0/2) &= \sup\{|f(s) - f(t)| \mid s, t \in (y - \delta_0/2, y + \delta_0/2)\} \\
&\leqslant \sup\{|f(s) - f(t)| \mid s, t \in (x_0 - \delta_0, x_0 + \delta_0)\} \\
&= \phi(x_0, \delta_0) < \alpha
\end{aligned}$$

Therefore $y \in V_\alpha$ for all $y \in B$, implying V_α is open in R^1. Hence ϕ is upper semicontinuous.

f is continuous at x iff $\phi(x) = 0$: First Suppose f is continuous at x. Therefore, for $\epsilon > 0$, there exists $\delta_0 > 0$, such that $|f(x) - f(y)| < \epsilon/2$ whenever $|x - y| < \delta_0$. But then $\sup\{|f(s) - f(t)| \mid s, t \in (x - \delta_0, x + \delta_0) < \epsilon\}$, i.e. $\phi(x, \delta_0) < \epsilon$. And therefore $\phi(x) \leqslant \phi(x, \delta_0) < \epsilon$. Make $\epsilon \to 0$ to conclude $\phi(x) = 0$.

Conversely, suppose $\phi(x) = 0$. Therefore $\lim_{\delta \to 0} \phi(x, \delta) = 0$. Therefore for $\epsilon > 0$, there exists δ_0, such that $\phi(x, \delta_0) < \epsilon$. But that means $\sup\{|f(s) - f(y)| \mid s, y \in (x - \delta_0, x + \delta_0)\} < \epsilon$. Take $s = x$, to get $|f(x) - f(y)| < \epsilon$, whenever $y \in (x - \delta_0, x + \delta_0)$, that is f is continuous at x.

Set of points of continuity is G_δ: Since f is continuous at x if and only if $\phi(x) = 0$, therefore the set of points of continuity of f is $\{x \mid \phi(x) = 0\}$. But

$$\{x \mid \phi(x) = 0\} = \bigcap_{n=1}^{\infty} \left\{ x \mid \phi(x) < \frac{1}{n} \right\}$$

Also since ϕ is upper semicontinuous, therefore each $\{x \mid \phi(x) < \frac{1}{n}\}$ is an open set. Hence $\{x \mid \phi(x) = 0\}$ is a G_δ.

Formulation for general topological spaces:

3 Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : y \in E\}.$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_E(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

Solution: For $x, y \in X$, we have

$$\begin{aligned} \rho_E(x) &\leqslant \rho(x, e) \text{ for all } e \in E \\ &\leqslant \rho(x, y) + \rho(y, e) \text{ for all } e \in E \\ \rho_E(x) - \rho(x, y) &\leqslant \rho(y, e) \text{ for all } e \in E \\ \text{Therefore, } \rho_E(x) - \rho(x, y) &\leqslant \rho_E(y) \\ \text{or } \rho_E(x) - \rho_E(y) &\leqslant \rho(x, y) \end{aligned}$$

Changing x with y, we get $|\rho_E(x) - \rho_E(y)| \leqslant \rho(x, y)$. So for $\epsilon > 0$, we chose $\delta = \epsilon$, and have $|\rho_E(x) - \rho_E(y)| \leqslant \rho(x, y) < \delta = \epsilon$, whenever $\rho(x, y) < \delta$. Hence ρ_E is uniformly continuous.

For A, B disjoint nonempty closed sets of X, and

$$f(x) = \frac{\rho_E(x)}{\rho_A(x) + \rho_B(x)}$$

we have $f(a) = 0$ for all $a \in A$; and $f(b) = 1$ for all $b \in B$. So for given K compact and V open set containing K, take $A = V^c$ and $B = K$, to get the desired function $K \prec f \prec V$ in Urysohn's lemma (2.12). ∎

4 Examine the proof of the Reisz theorem and prove the following two statements:
(a) If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if E_1 and E_2 are not in $\mathfrak{M}$.
(b) If $E \in \mathfrak{M}_F$, then $E = N \cup K_1 \cup K_2 \cup \cdots$, where K_i is a disjoint countable collection of compact sets and $\mu(N) = 0$.
Solution: (a)

CHAPTER

FOUR

ELEMENTARY HILBERT SPACE THEORY

1 If M is a closed subspace of H, prove that $M = (M^\perp)^\perp$. Is there a similar true statement for subspaces M which are not necessarily closed?

SOLUTION: We first show $M \subset (M^\perp)^\perp$. Since $x \in M \Rightarrow x \perp M^\perp \Rightarrow x \in (M^\perp)^\perp$. Hence $M \subset (M^\perp)^\perp$.

Next we will show $(M^\perp)^\perp \subset M$. Since M is a closed subspace of H, therefore $H = M \oplus M^\perp$ (Theorem 4.11). So if $x \in (M^\perp)^\perp$, then $x = y + z$ for some $y \in M$ and $z \in M^\perp$. Consider $z = x - y$. Since $x \in (M^\perp)^\perp$ and $y \in M \subset (M^\perp)^\perp$; combining with the fact that $(M^\perp)^\perp$ is a subspace, we get $x - y = z \in (M^\perp)^\perp$. But that would mean $z \perp M^\perp$. Also we started with $z \in M^\perp$. Together it implies $z = 0$. Therefore $x = y \in M$. And so $(M^\perp)^\perp \subset M$.

So for closed subspace M, we have $M = (M^\perp)^\perp$.

Next suppose M is a subspace which may not be closed. So we have $\overline{M} = (\overline{M}^\perp)^\perp$. We further simplify the expression by showing $\overline{M}^\perp = M^\perp$. If $x \perp \overline{M}$, then it would imply $x \perp M$, therefore $\overline{M}^\perp \subset M^\perp$. For reverse inclusion, consider $x \in M^\perp$, we will show $x \in \overline{M}^\perp$. Let $m \in \overline{M}$, therefore there exists a sequence $\{m_i\}$ in M such that $m_i \to m$. Since $x \in M^\perp$, therefore $\langle x, m_i \rangle = 0$ for all i. Continuity of inner product (Theorem 4.6) would imply $\langle x, m \rangle = 0$ too. Therefore $\langle x, m \rangle = 0$ for all $m \in \overline{M}$. So $x \in \overline{M}^\perp$. Hence for any subspace M, we have $\overline{M} = (M^\perp)^\perp$. ∎

2 Let $\{x_n \mid n = 1, 2, 3, \cdots\}$ be a linearly independent set of vectors in H. Show that the following construction yields an orthonormal set $\{u_n\}$ such that $\{x_1, \cdots, x_N\}$ and $\{u_1, \cdots, u_N\}$ have the same span for all N.
Put $u_1 = x_1/\|x_1\|$. Having $u_1, \cdots, u_{n-1}$ define

$$v_n = x_n - \sum_{i=1}^{n-1} \langle x_n, u_i \rangle u_i, \quad u_n = v_n/\|v_n\|.$$

SOLUTION: We need to show $\{u_1, \cdots, u_n\}$ is orthonormal and the span of $\{u_1, \cdots, u_n\}$ is equal to the span of $\{x_1, \cdots, x_n\}$ for all $n \in \mathbb{N}$. We show it by induction on n.

For $n = 1$, we have $\{u_1\}$ othonormal set and also $\mathrm{span}(u_1) = \mathrm{span}(x_1)$. Therefore the result is true for $n = 1$.

Suppose the result is true for $n = N-1$, that is $\{u_1, \cdots, u_{N-1}\}$ is an orthonormal set and $\mathrm{span}(u_1, \cdots, u_{N-1}) = \mathrm{span}(x_1, \cdots, x_{N-1})$. We need to show $\{u_1, \cdots, u_N\}$ is an orthonormal set and $\mathrm{span}(u_1, \cdots, u_N) = \mathrm{span}(x_1, \cdots, x_N)$.

To show $\{u_1, \cdots, u_N\}$ is orthonormal, it will suffice to show $\langle u_N, u_i \rangle = 0$ for $i = 1$ to $N-1$, as $\{u_1, \cdots, u_{N-1}\}$ is already orthonormal. Also

$$\begin{aligned}
\langle u_N, u_i \rangle &= \frac{1}{\|v_N\|} \langle v_N, u_i \rangle \\
&= \frac{1}{\|v_N\|} \langle x_N - \sum_{j=1}^{N-1} \langle x_N, u_j \rangle u_j, u_i \rangle \\
&= \langle x_N, u_i \rangle - \langle x_N, u_i \rangle \langle u_i, u_i \rangle \\
&= 0
\end{aligned}$$

Hence $\{u_1, \cdots, u_N\}$ is orthonormal.

Next we have

$$\begin{aligned}
x \in \mathrm{span}(x_1, \cdots, x_{N-1}, x_N) &\iff x \in \mathrm{span}(u_1, \cdots, u_{N-1}, x_N) \\
&\iff x \in \mathrm{span}(u_1, \cdots, u_{N-1}, v_N) \\
&\iff x \in \mathrm{span}(u_1, \cdots, u_{N-1}, u_N)
\end{aligned}$$

So the result is true of $n = N$. Hence the result is true for all $n \in \mathbb{N}$ ∎

3 Show that $L^p(T)$ is separable if $1 \leq p < \infty$, but that $L^\infty(T)$ is not separable.

SOLUTION: $L^p(T)$ for $1 \leq p < \infty$: Let $P(T)$ denotes the subspace of trigonometric polynomials in $L^p(T)$. Easy to check $P(T)$ is separable using the fact that $\mathbb{Q} + i\mathbb{Q}$ is countable and dense in $\mathbb{C}$. Also one can show that $P(T)$ is dense in $C(T)$ with respect to $\| \|_p$ norm using the argument given in 4.24 (or using Fejèr theorem). Also $C(T)$ is dense in $L^p(T)$ (Theorem 3.14). So $P(T)$ is separable and dense in $L^p(T)$, implying $L^p(T)$ is separable too.

$L^\infty(T)$: Since $L^\infty(T)$ can be identified as $L^\infty([0, 2\pi])$, we will show $L^\infty([0, 2\pi])$ is not separable. Consider

$$S = \{f \in L^\infty([0, 2\pi]) \mid f = \chi_{[0,r]} \text{ for } 0 < r < 2\pi\}$$

So S is an uncountable subset of $L^\infty([0, 2\pi])$. Also if $f, g \in S$ with $f \neq g$, then $\|f - g\|_\infty = 1$. Suppose $L^\infty([0, 2\pi])$ is separable. Therefore there exists a countable dense set, say M in $L^\infty([0, 2\pi])$. But then $\cup_{m \in M} B_{0.4}(m)$ must contain $L^\infty([0, 2\pi])$, where $B_{0.4}(m)$ denotes an open ball of radius 0.4 around m. Since each $B_{0.4}(m)$ contains at most one element of S, therefore $\cup_{m \in M} B_{0.4}(m)$ contains at most countable elements of S. S being uncountable, so S is not a subset of $\cup_{m \in M} B_{0.4}(m)$, a contradiction. Therefore $L^\infty([0, 2\pi])$ is not separable. ∎

4 Show that H is separable if and only if H contains a maximal orthonormal system which is at most countable.

SOLUTION: First suppose H is separable, then by **Exercise 2**, H has at most countable maximal orthonormal set.

Conversely, suppose H is a Hilbert space with countable maximal orthonormal set E. Therefore $E = \{u_1, u_2, \cdots\}$ for some $u_i \in H$. Define

$$S = \{\sum_{\text{finite}} \alpha_i u_i \mid \alpha_i \in \mathbb{Q} + i\mathbb{Q} \;\&\; u_i \in E\}$$

Clearly S has countable elements. So if we show $\bar{S} = H$, we are done.

Let $x \in H$, therefore $x = \sum_{i=1}^{\infty} \alpha_i u_i$, with $\sum_{i=1}^{\infty} |\alpha_i|^2 = \|x\|^2$. Let some $\epsilon > 0$, therefore there exists $n \in \mathbb{N}$ such that $\sum_{i=n+1}^{\infty} |\alpha_i|^2 < \epsilon^2/4$. Define $\bar{x} = \sum_{i=1}^{n} \alpha_i u_i$. Therefore $\|x - \bar{x}\|^2 = \sum_{i=n+1}^{\infty} |\alpha_i|^2 < \epsilon^2/4$. Also define $y = \sum_{i=1}^{n} \beta_i u_i$ where $\beta_i \in \mathbb{Q} + i\mathbb{Q}$ such that $\|\bar{x} - y\|^2 = \sum_{i=1}^{n} |\alpha_i - \beta_i|^2 < \epsilon^2/4$; such construction is possible since $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$. So $y \in S$, and $\|x - y\| = \|x - \bar{x} + \bar{x} - y\| \leqslant \|x - \bar{x}\| + \|\bar{x} - y\| < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore S is dense in H. Same proof will work is H has finite maximal orthonormal system. Hence H is separable.

REMARKS: Another way of showing that in a Hilbert space separability implies that space has at most countable orthonormal system, is through contradiction. Suppose E be the uncountable maximal orthonormal system. If $u_1, u_2 \in E$, then $\|u_1 - u_2\|^2 = \|u_1\|^2 + \|u_2\|^2 = 2$. Therefore the collection $\{B(u, 0.5) \mid u \in E\}$ is uncountable. Also each element of this collection is disjoint; showing that H cannot have a countable dense subset, hence not separable, a contradiction. ∎

5 If $M = \{x \mid Lx = 0\}$, where L is a continuous linear functional on H, prove that $M^{\perp}$ is vector space of dimension 1 (unless $M = H$).

SOLUTION: If $M = H$, then $L = 0$, therefore we assume $L \neq H$. Its easy to check that M is a closed subspace of H. Also using Theorem 4.12 (Riesz representation theorem), we have $L(x) = \langle x, x_0 \rangle$ for some $x_0 \in H$ with $x_0 \neq 0$, since we have assumed $L \neq 0$. So we have

$$\begin{aligned} M &= \{x \in H \mid \langle x, x_0 \rangle = 0\} \\ M &= x_0^{\perp} \\ M^{\perp} &= (x_0^{\perp})^{\perp} \\ M^{\perp} &= \overline{\text{Span}\{x_0\}} \text{ (Using } \textbf{Exercise 1}\text{)} \end{aligned}$$

Therefore $M^{\perp}$ is vector space of dimension 1. ∎

6 Let $\{u_n\}(n = 1, 2, 3 \ldots)$ be an orthonormal set in H. Show that this gives an example of closed and bounded set which is not compact. Let Q be the set of all $x \in H$ of the form

$$x = \sum_{1}^{\infty} c_n u_n \qquad \left(\text{where } |c_n| \leq \frac{1}{n}\right).$$

Prove that Q is compact. (Q is called the Hilbert cube.)
More generally, let $\{\delta_n\}$ be a sequence of positive numbers, and let S be the set of all $x \in H$ of the form

$$x = \sum_{1}^{\infty} c_n u_n \qquad (\text{where } |c_n| \leq \delta_n).$$

Prove that S is compact if and only if $\sum_1^\infty \delta_n^2 < \infty$.
Prove that H is not locally compact.

SOLUTION:

7 Suppose $\{a_n\}$ is a sequence of positive numbers such that $\sum a_n b_n < \infty$ whenever $b_n \geq 0$ and $\sum b_n^2 < \infty$. Prove that $a_n^2 < \infty$.

SOLUTION: One way is follow the suggestion, but we will give an alternate method using Banach-Steinhaus theorem (**Theorem 5.8**).

For $n \in \mathbb{N}$, define $\Lambda_n : l^2(\mathbb{R}) \longrightarrow \mathbb{R}$ such that $\Lambda_n(x) = \sum_{i=1}^n a_i x_i$, where $x = (x_1, x_2, \cdots)$.

Λ_n is linear for all n, is easy to check.

For $x \in l^2(\mathbb{R})$, we have

$$|\Lambda_n(x)| = |\sum_{i=1}^{n} a_i x_i| \leqslant \sum_{i=1}^{n} a_i |x_i| \leqslant \sum_{i=1}^{\infty} a_i |x_i| < \infty$$

since $\sum |b_i|^2 < \infty$ and hypothesis says whenever $\sum |b_i|^2 < \infty$ implies $\sum a_i b_i < \infty$. Therefore $|\Lambda_n(x)|$ is bounded for all n. And this is true for all $x \in l^2(\mathbb{R})$ too. Also Baire's Category theorem (see **Section 5.7**) implies $l^2(\mathbb{R})$ is of second category, since $l^2(\mathbb{R})$ is complete. Invoking Banach-Steinhaus theorem on collection $\{\Lambda_n\}$, we get $\|\Lambda_n\| < M$ for some $M > 0$. Also we have

$$\begin{aligned}
\|\Lambda_n\| &= \sup_{x\neq 0} \frac{|\sum_{i=1}^n a_i x_i|}{(\sum_{i=1}^\infty |x_i|^2)^{1/2}} \\
&\geqslant \frac{\sum_{i=1}^n a_i^2}{(\sum_{i=1}^n a_i^2)^{1/2}} \quad \text{(By taking } x = (a_1, \cdots, a_n, 0, 0, \cdots)) \\
&= (\sum_{i=1}^n a_i^2)^{1/2}
\end{aligned}$$

So we have

$$\sum_{i=1}^n a_i^2 \leqslant \|\Lambda_n\|^2 < M^2 \quad \forall\, n \in \mathbb{N}$$

Taking $n \to \infty$, we get $\sum_{i=1}^\infty a_i^2 < M^2 < \infty$ ∎

8 If H_1 and H_2 are two Hilbert spaces, prove that one of them is isomorphic to a subspace of the other. (Note that every closed subspace of a Hilbert space is a Hilbert space.)

SOLUTION:

9 If $A \subset [0, 2\pi]$ and A is measurable, prove that

$$\lim_{n\to\infty} \int_A \cos nx\, dx = \lim_{n\to\infty} \int_A \sin nx\, dx = 0.$$

SOLUTION: We know that $\{u_n \mid n \in \mathbb{Z}\}$ is maximal orthomormal set for $L^2(\mathbb{T})$, where $u_n(x) = e^{inx}$. Now consider χ_A, characteristic function of A. Since

$$\sum_{n=-\infty}^{\infty} |\langle u_n, \chi_A\rangle|^2 = \|\chi_A\|^2 = m(A) < \infty$$

Therefore $\lim_{|n|\to\infty} |\langle u_n, \chi_A\rangle| = 0$. So we have

$$\lim_{n\to\infty} \left\langle \frac{u_n + u_{-n}}{2}, \chi_A \right\rangle = \lim_{n\to\infty} \left\langle \frac{u_n - u_{-n}}{2i}, \chi_A \right\rangle = 0$$

which is nothing but

$$\lim_{n\to\infty}\int_A \cos nx\,dx = \lim_{n\to\infty}\int_A \sin nx\,dx = 0. \qquad \blacksquare$$

10 Let $n_1 < n_2 < n_3 \cdots$ be positive integers, and let E be the set of all $x \in [0, 2\pi]$ at which $\{\sin n_k x\}$ converges. Prove that $m(E) = 0$.

SOLUTION:

11 Find a nonempty closed set in $L^2(\mathbb{T})$ that contains no element of smallest norm.

SOLUTION: Let $E = \{\left(1 + \frac{1}{n}\right) u_n \mid n \in \mathbb{N}\}$. E is closed since for $a, b \in E$, we have $\|a - b\| > \sqrt{2}$, hence no limit point. Also $\inf_{a\in E} \|a\| = 1$ but is not achieved by any element of E. $\blacksquare$

12 The constants c_k in Sec. 4.24 were shown to be such that $k^{-1}c_k$ is bounded. Estimate the relevant integral more precisely and show that

$$0 < \lim_{k\to\infty} k^{-1/2} c_k < \infty.$$

SOLUTION:

13 Suppose f is a continuous function on R^1, with period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)\,dt$$

for every irrational real number α.

SOLUTION: Note that this problem is famous *Wiel Equidistribution theorem*. As the Hint goes, we first check the equality for $\{e^{2\pi i k x}\}$ where $k \in \mathbb{Z}$.

When $k = 0$, we have $f(x) = 1$. So we have $\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = 1$ for all N. Therefore

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = 1 = \int_0^1 f(t)\,dt$$

When $k \neq 0$, we have $e^{2\pi i k\alpha} \neq 1$, since α is an irrational. So we have

$$\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \frac{1}{N}\frac{e^{2\pi i k\alpha}(1 - e^{2\pi i N k\alpha})}{(1 - e^{2\pi i k\alpha})} \longrightarrow 0 \text{ as } N \to \infty$$

Also $\int_0^1 f(t)\,dt = 0$ since $k \neq 0$. Hence the identity holds for the collection $\{e^{2\pi i k x}\}_{k\in\mathbb{Z}}$. If the identity holds for f and g then it also holds for $af + bg$, where $a, b \in \mathbb{C}$; and hence the identity holds for all trigonometric polynomials.

Let $\epsilon > 0$, and f be a continuous function of period 1. **Theorem 4.25** implies that there will exists a trigonometric polynomial p such that $\|f - p\|_\infty < \epsilon/3$. Also for large N, we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} p(n\alpha) - \int_0^1 p(t)\,dt\right| < \epsilon/3$$

And therefore

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) - \int_0^1 f(t)\,dt\right| &\leqslant \frac{1}{N}\sum_{n=1}^{N} |f(n\alpha) - p(n\alpha)| \\ &\quad + \left|\frac{1}{N}\sum_{n=1}^{N} p(n\alpha) - \int_0^1 p(t)\,dt\right| \\ &\quad + \int_0^1 |p(t) - f(t)|\,dt \\ &\leqslant \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon\end{aligned}$$

Hence

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)\,dt \text{ for all continuous functions of period 1} \qquad \blacksquare$$

14

SOLUTION:

CHAPTER

NINE

FOURIER TRANSFORMS

1 Suppose $f \in L^1$, $f > 0$. Prove that $|\hat{f}(y)| < \hat{f}(0)$ for every $y \neq 0$.
Solution: We have

$$\begin{aligned} |\hat{f}(y)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} f(x) e^{-ixy} \, dx \right| \\ &\leqslant \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} |f(x)| \, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} f(x) \, dx = \hat{f}(0) \end{aligned}$$

For strict inequality, suppose $\hat{f}(y) = \hat{f}(0)$ for some $y \neq 0$. So

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} f(x) e^{-ixy} \, dx &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} f(x) \, dx \\ \text{that is } \int_{\infty}^{\infty} f(x)(e^{-ixy} - 1) \, dx &= 0 \\ \text{comparing real part, we get } \int_{\infty}^{\infty} f(x)(\cos(xy) - 1) \, dx &= 0 \end{aligned}$$

Therefore, $\cos(xy) = 1$ ae, which is only possible if $y = 0$, a contradiction. Hence the strict inequality. ∎

1 Compute the Fourier transform of the characteristic function of an interval. For $n = 1, 2, 3 \cdots$, let g_n be the characteristic function of $[-n, n]$, let h be the characteristic function of $[-1, 1]$, and compute $g_n * h$ explicitly. (The graph is piecewise linear.) Show that $g_n * h$ is the Fourier transform of a function $f_n \in L^1$; except for a multiplicative constant,

$$f_n(x) = \frac{\sin(x)\sin(nx)}{x^2}$$

Show that $\|f_n\|_1 \to \infty$ and conclude that the mapping $f \to \hat{f}$ maps L^1 into a proper subset of C_0.
Show, however, that the range of this mapping is dense in C_0.
Solution: